Stat 512 — Take home exam III (due on July 22nd)

1. If $Y \sim F_{v_1,v_2}$, then prove: $\dfrac{\left(\frac{v_1}{v_2}\right)Y}{1+\left(\frac{v_1}{v_2}\right)Y} \sim Beta(\frac{v_1}{2}, \frac{v_2}{2})$. (10 pts)

2. Assume now that the amount of fill dispensed by the bottling machine is exponentially distributed with $\beta = 2$, that is $Y_1, \ldots, Y_n \sim Exp(2)$.

   a. What is the asymptotic distribution of $\bar{Y}$ when $n = 3$? Using R or standard normal table to find out $P|\bar{Y} - 2| \leq 3$. (10 pts)
   (Hint: Assume CLT is appropriate here; asymptotic distribution means the limiting distribution, that is, the distribution when $n \to \infty$).

   b. Obtain the exact distribution of $\bar{Y}$ using MGF technique. Is this a common distribution that you've seen before? (10 pts)

3. The flow of water through soil depends on, among other things, the porosity (volume proportion of voids) of the soil. To compare two types of sandy soil, $n_1 = 50$ measurements are to be taken on the porosity of soil A and $n_2 = 100$ measurements are to be taken on the porosity of soil B. Assume that $\sigma_1^2 = 0.01$ and $\sigma_2^2 = 0.02$. Population means for the two types of soil are μ_1, μ_2. Denote the samples for the soil A be $X_1, \ldots, X_{50}$, the samples for the soil B be $Y_1, \ldots, Y_{100}$.

   a. What is the asymptotic distribution for $\bar{X} - \bar{Y}$? (10 pts)

   b. What is the probability that sample variance of soil A is at least twice as large as the sample variance of soil B? (Use R to help you find the answer. The command is: *pf(q,df1,df2)*, for more details; type *?pf* in R) (10 pts)

4. Suppose Y follows a binomial distribution with parameter n and p, then $\dfrac{Y}{n}$ is an unbiased estimator of p. Now, we want to estimate the variance of Y which is $np(1-p)$, we proposed a new estimator: $\widehat{p} = n\left(\frac{Y}{n}\right)\left(1 - \frac{Y}{n}\right)$.

   a. Show that $\widehat{p}$ is biased for the variance of Y. (10 pts)

   b. Adjust the estimator so that the new estimator is an unbiased estimator of $np(1-p)$. Hint: "Adjust" means you need to find a function of $\widehat{p}$ such that the new function is unbiased. Remember that an estimator cannot involve unknown parameters. (10 pts)

5. Suppose that $Y_1, \ldots, Y_n$ denote a random sample of size n from a population with an exponential distribution $exp(\beta)$. $Y_{(1)} = min(Y_1, \ldots, Y_n)$ denotes the smallest-order ststistic.

   a. Show that $\hat{\beta}_1 = nY_{(1)}$ is an unbiased estimator for β. (10 pts)

   b. Show that $\hat{\beta}_2 = \bar{Y}$ is also an unbiased estimator for β. (10 pts)

   c. Compare the variance of the two estimators. (10 pts)

Extra credit: $Y_1, \ldots, Y_n$ are i.i.d. from $U(\theta, \theta + 1)$. Define $\hat{\theta}_1 = Y_{(1)}$. Adjust $\hat{\theta}_1$ so that it becomes an unbiased estimator for θ and compute the associated variance for the adjusted estimator. (10 pts)